Hamiltonian Mechanics

- Yet Another Description of mechanics (Yay!)
  - Newton: all types of forces, particle trajectories in $\mathbb{R}^3$
    only valid in inertial frames, 2nd Order ODEs
  - Lagrange: configuration manifold; eliminates constraints; valid in all coordinate frames; visualization less trivial

    2nd Order ODEs

- Hamilton:
  - doesn't really add new capabilities, but brings out mathem. geometrical, structure of mechanics, allowing us to generalize these structures to Stat Mech & QM
  - we'll see that it leads to first-order ODEs – we have implicitly used Hamilton EoMs when we implemented mechanical systems numerically.

Lagrangian vars:

$$q_j, \dot{q}_j, t \implies q_j, p_j, t$$

(q,t) configuration space $\qquad$ (q,p) phase space

where

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \text{canonical momentum}}$$

Note: need not be a mechanical momentum!

- q_j, p_j are independent variables! Go from n second-order ODEs for q_j with $2n$ initial conds. to $2n$ first-order ODEs with $2n$ initial conds. → cf. numerical implementation!
- p_j are the conserved quantities when variables are cyclical.

- We cannot just solve

$$p_k(q_j, \dot{q}_j, t)$$

for $\dot{q}_j$ and use this to eliminate $\dot{q}_j$ from L, because this might make the transformation from Lagrange to Hamilton picture non-unique and non-invertible – so we would get the wrong EoM's!

Legendre transformations

Assume you have $f(x,y)$ but $z = \frac{\partial f}{\partial y}$ is more interesting, e.g. because it's measurable more easily, conserved, etc. (e.g. $L(q,\dot q)$ and $p = \frac{\partial L}{\partial \dot q}$ instead of $\dot q$)

⇒ Want z as an <u>independent</u> variable and replace y by z.

Naive: ~~invert~~ solve $z = \frac{\partial f}{\partial y}$ for $y(x,z)$ ⇝ $f(x, y(x,z)) = \tilde f(x,z)$

Problems: not unique, lost information!
(not invertible)

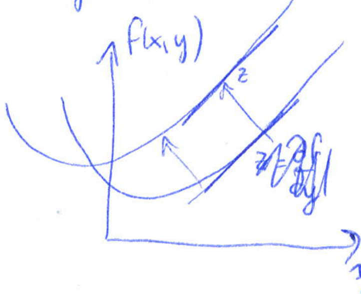
<u>Ex.:</u> $f(x,y) = x^2 + (y-a)^2$ ← <u>family</u> of functions, with different a

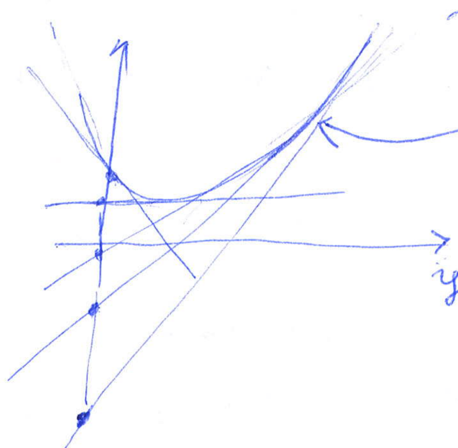
$\Rightarrow z = \frac{\partial f}{\partial y} = 2(y-a) \Rightarrow y(x,z) = \frac{z}{2} + a$

$\Rightarrow \tilde f(x,z) = f(x, y(x,z)) = x^2 + \frac{z^2}{4}$ → a is <u>gone!</u>

⇒ Get the same $\tilde f(x,z)$ for ~~different~~ functions with different values of a!

Why? z is the slope of f!





sign for convenience

$$\boxed{\underbrace{\frac{\partial f}{\partial y}}_{z \text{ as}} y - g(x,z) = f(x,y)}$$

~~can shift $f(x,y)$ without changing slope~~

knowing $f(x,y)$ does not specify function everywhere.

intercept & slope works for <u>convex</u> functions of replaced var! ~~& g qualifies since it's concave~~

~~Denote intercept by $g(x,z)$ ← all fixed, ← slope at y~~

(i) $z = \frac{\partial f}{\partial y} \Rightarrow y(x,z)$

(ii) Define Legendre transform as

$$g(x,z) = z\,y(x,z) - f(x, y(x,z))$$

→ Process can be readily inverted. For our example:

$$f(x,y) = x^2 + (y-a)^2 \qquad z = \frac{\partial f}{\partial y} = 2(y-a) \Rightarrow y(x,z) = \frac{z}{2} + a \quad \text{(see above)}$$

$$\rightsquigarrow g(x,z) = z\left(\frac{z}{2}+a\right) - \left(x^2 + \frac{z^2}{4}\right) = \frac{z^2}{2} + az - x^2 - \frac{z^2}{4} = \frac{z^2}{4} + az - x^2$$

$$\frac{\partial g}{\partial z} = z/2 + a \left(= \frac{\partial f}{\partial y} \right)$$

$$\Rightarrow f(x,y) = y \cdot 2(y-a) - g(x, z(x,y)) = 2(y-a)y - (y-a)^2 - 2a(y-a) + x^2$$
$$= 2(y-a)^2 - (y-a)^2 + x^2$$
$$= (y-a)^2 + x^2 \;\checkmark$$

- The Legendre transformation readily extends to multiple variables ... just apply the process while keeping all but the current var. fixed.
- Here: y, z <u>active</u> vars, x <u>passive</u> var.

- We can apply the Legendre trafo to the Lagrangian.
  - replace generalized velocities $\dot{q}_j \to p_j$ canonical momenta (see above) (since they may be conserved ...)
  - $L(q_j, \dot{q}_j, t)$ is <u>convex</u> if $\dot{q}_j, p_j$ are active – if $V = V(q)$ – $\dot{q}_j$ appear in kinetic energy quadratically (but we'll see that the procedure works for $U(q,\dot{q})$) ↓ linear fcts are concave and convex

Legendre trafo of $L(q_j, \dot{q}_j, t)$:

<u>Hamiltonian</u> (now properly b/c given as fct. of q, p)

$$H(q_j, p_j, t) = \sum_k p_k \dot{q}_k(q_j, p_j, t) - \tilde{L}(q_j, p_j, t)$$

$$L(q_j, \dot{q}_j, t) = \sum_k \dot{q}_k p_k(q_j, \dot{q}_j, t) - \tilde{H}(q_j, \dot{q}_j, t) \;; \quad \boxed{\dot{q}_k = \frac{\partial H}{\partial p_k}}$$

Hamilton's Eqns.

We already saw

$$\dot q_j = \frac{\partial H}{\partial p_j}$$

from Legendre trafo.

Consider differential:

$$dH = \sum_j \left(\dot q_j \, dp_j + \cancel{p_j \, d\dot q_j} \right) - \left(\sum_j \left(\frac{\partial L}{\partial q_j} dq_j + \cancel{\underbrace{\frac{\partial L}{\partial \dot q_j}}_{= p_j} d\dot q_j} \right) + \frac{\partial L}{\partial t} dt \right)$$

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt} \frac{\partial L}{\partial \dot q_j} = \frac{d}{dt} p_j = \dot p_j$$

$$\Rightarrow \quad dH = \sum_j \left(\dot q_j \, dp_j - \dot p_j \, dq_j \right) - \frac{\partial L}{\partial t} dt \qquad \left(\Rightarrow H = H(q_j, p_j, t) \right)$$

We also have

$$dH = \sum_j \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt$$

$\Rightarrow$ ~~do~~ comparison of coeffs.

$$\boxed{\dot q_j = \frac{\partial H}{\partial p_j} \qquad -\dot p_j = \frac{\partial H}{\partial q_j} , \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

General procedure:

(I) Choose q_j ~~[illegible]~~, construct $L(q_j, \dot q_j, t)$

(II) Construct canonical momenta ~~[illegible]~~ solve for ~~[illegible]~~ $\dot q_k(q_j, p_j, t)$

(III) Construct $H(q_j, p_j, t)$

(IV) Write down EoM's & solve

Examples Do Harmonic Oscillator first!

(2) Particle in central potential:

$$L = T - V = \frac{m}{2}\left(\dot r^2 + r^2 \dot\phi^2 \right) - V(r)$$

$$\Rightarrow \quad p_r = \frac{\partial L}{\partial \dot r} = m\dot r , \quad p_\phi = m r^2 \dot\phi \;\; (= \ell)$$

$$\Rightarrow \quad \dot r = \frac{p_r}{m} , \quad \dot\phi = \frac{p_\phi}{m r^2}$$

$$\Rightarrow H(p_r, p_\phi, r, \phi) = \frac{p_r^2}{m} + \frac{p_\phi^2}{m r^2} - \frac{p_r^2}{2m} - \frac{p_\phi^2}{2mr^2} + V(r)$$

$$= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r} ; \quad \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} = \dot{\phi}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r} \quad \Rightarrow \quad m\ddot{r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r} = \frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \Rightarrow p_\phi = \text{const.} \Rightarrow m r^2 \dot{\phi} = \text{const.} = l$$